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# THE CONSTRUCTION OF A MODEL OF FREQUENCY-INDEPENDENT DAMPING USING THE AMPLITUDE CHARACTERISTIC OF THE ABSORPTION COEFFICIENT†

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Relations are obtained which connect the stresses and strains accompanying oscillations with the absorption coefficient and a procedure for modelling the frequency-independent scattering of energy is constructed based on them. Account is taken of the fact that damping lowers the resonance frequency and the accumulation of residual stresses and strains is not observed during oscillation. The models obtained using this procedure can be employed for any changes, including unsteady changes, in the load with respect to time. The case of a power dependence of the absorption coefficient on the amplitude, which is important in practice, is investigated in detail. The forced, transverse oscillations of a rod are considered as an example. Asymptotic and numerical solutions of the problem are obtained in the single-mode approximation. © 2003 Elsevier Science Ltd. All rights reserved.

The frequency-independent scattering of energy is associated above all with structural damping and, at a level of stresses which is commensurate with the fatigue limit, also with energy losses in the material. Usually, this is the basic type of energy scattering in structures and installations. Different approximations of the hysteresis loop are used in order to allow for it in the case of steady oscillations. The width of the loop is usually small and the nature of the oscillations is determined primarily by the area of the loop and not by its shape. In this case, numerous models are shown to be equivalent to a significant degree and can often be replaced by a model with an elliptic hysteresis loop, which is convenient for carrying out calculations. In practice, the method of complex stiffnesses is also applicable to steady harmonic oscillations or oscillations which are close to them [1–3].

When transitional and wave processes are considered, the above-mentioned methods for describing frequency-independent damping often turn out to be ineffective. The use of some of them leads to the appearance of residual strains. In other models, the graphs of the stresses and strains have sections where the stresses undergo a discontinuity [1]. The existence of discontinuities leads to appearance of domains, in the object under investigation, with an infinite propagation velocity of perturbations. The development of models of frequency-independent damping, which are free from these drawbacks remains an urgent problem.

## 1. THE PROCEDURE FOR CONSTRUCTING THE MODEL

The basis of the procedure is the determination of the absorption coefficient as the ratio of the energy scattered during a cycle of oscillations to the maximum potential energy during the course of a cycle [1]. First of all, we take into account that the modulus of elasticity, which we determine for an amplitude of the oscillations tending to zero, is the maximum tangent modulus. As the amplitude becomes larger, the increasing frequency-independent damping lowers the frequency of the natural oscillations, the propagation velocity of perturbations and the tangent modulus, which determines their magnitude. We shall assume that the scattering of energy occurs in a like manner during the loading and unloading stages and that it is not accompanied by the appearance of residual stresses and strains. By virtue of this last fact, the hysteresis loop, both during the complete removal of the load as well as during sign-variable loading, must pass through the origin of coordinates in the plane of the strain–stress parameters.

The constraints imposed on the shape of the hysteresis loop enable us to write the following expression for the absorption coefficient in the case of steady oscillations over a symmetric cycle

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$$\psi = 4\left(1 - \frac{\sigma\varepsilon}{2P}\right), \quad P = \int_0^{\varepsilon} \sigma(\varepsilon_1) d\varepsilon_1, \quad \varepsilon > 0 \tag{1.1}$$

where $\psi$ is the absorption coefficient $\sigma$ and $\varepsilon$ are the maximum stresses and strains, attained at the loading stage, and $P$ is the largest value of the potential energy in the cycle.

It is obvious that $\sigma(\varepsilon) = P'$ (a derivative with respect to $\varepsilon$ is indicated with a prime) and the expression for $P$ can be written in the form of a differential equation with separable variables in the case of a specified coefficient $\psi(\varepsilon)$

$$\varepsilon P' = (2 - \psi/2)P \tag{1.2}$$

Its solution has the form

$$P = C\varepsilon^2 \exp\left(-\frac{1}{2}\int_0^{\varepsilon} \frac{\psi(\varepsilon_1)}{\varepsilon_1} d\varepsilon_1\right) \tag{1.3}$$

where $C$ is a constant, which is determined from the initial conditions for $P(\varepsilon)$.

When the variation in the absorption coefficient $\psi(\varepsilon)$ is known, formula (1.3) enables as to determine the form of the $\sigma - \varepsilon$ diagram at the loading stage. Later, we will refer the stresses to the magnitude of the modulus of elasticity. The requirements formulated earlier regarding the shape of the $\sigma - \varepsilon$ diagram impose the following constraints on the function $P(\varepsilon)$

$$0 \le P'' \le 1, \quad P''(0) = 1 \tag{1.4}$$

Account has been taken of the fact that $P'' = \sigma'$ is the current tangent modulus.

We will subsequently assume that the $\sigma - \varepsilon$ diagram has the same form for extension and compression and the second derivative $P''$ must therefore be an even function.

Note that the function $P'' = 1$, for example, corresponds to all above-mentioned conditions. On integrating it twice and taking account of the fact that $P(0) = 0$ and $\sigma(0) = 0$, we obtain $P = \varepsilon^2/2$. It is well-known that this form of the function $P(\varepsilon)$ corresponds to absolutely elastic strain. After substituting this expression into Eq. (1.2), we also obtain that $\psi = 0$, that is, the oscillations are not accompanied by the scattering of energy. This result confirms the consistency of the initial propositions which have been adopted.

The experimental curves of the absorption coefficient against the amplitude are customarily approximated using a power function [2, 3]

$$\psi = A\varepsilon^{\alpha} \tag{1.5}$$

After substituting expression (1.5) into Eq. (1.3) and taking account of condition (1.4) at $\varepsilon = 0$, we obtain, when $\alpha \neq 0$

$$P = \frac{\varepsilon^2}{2}\exp\left(-\frac{\psi}{2\alpha}\right), \quad \sigma = \left(1 - \frac{\psi}{4}\right)\varepsilon \exp\left(-\frac{\psi}{2\alpha}\right) \tag{1.6}$$

When $\alpha = 0$, that is, the level of damping is independent of the amplitude, we have

$$P = C\varepsilon^{2-\theta}, \quad \sigma = C(2-\theta)\varepsilon^{1-\theta}, \quad \theta = A/2 \tag{1.7}$$

The derivative $\sigma'$, when $\alpha = 0$, has a singularity at the point $\varepsilon = 0$. This means that, in the case of very small oscillation amplitudes, the frequency-independent scattering of energy, at least according to the modelling technique which has been adopted, cannot remain constant.

When account is taken of unloading, it can be assumed [4] that

$$\begin{aligned} \sigma &= \left(1 - \frac{z}{4}\right)(\varepsilon - \varepsilon_m)\exp\left(-\frac{z}{2\alpha}\right) + \sigma_m, \quad z = A|\varepsilon - \varepsilon_m|^{\alpha} \\ \sigma' &= \left(1 - \frac{\alpha + 3}{4}z + \frac{z^2}{8}\right)\exp\left(-\frac{z}{2\alpha}\right) \end{aligned} \tag{1.8}$$

where $\varepsilon_m$ and $\sigma_m$ are the strain and stress at the instant of the last change from loading to unloading or vice versa. The values of the parameters at this instant are henceforth everywhere labelled with the subscript $m$.

The tangent modulus, according to conditions (1.4), cannot be negative. This assertion is analogous to the proposition concerning the convexity of the loading surface, which is adopted in the theory of plasticity [5]. An analysis of the expression for $\sigma'$, according to the last relation of (1.8), shows that the least value of $z$, down to which $\sigma' \geqslant 0$ at the loading stage, is given by the equality

$$z_* = \alpha + 3 - \sqrt{\alpha^2 + 6\alpha + 1}$$

Since Eqs (1.8) can only be used when $z < z_*$, this value of $z$ limits the largest value of the absorption coefficient which can be reproduced by the model in the case of a power form of $\psi(\varepsilon)$.

In practical calculations, it is usually more convenient to use an expansion of the expressions in a series in powers of $z$

$$\begin{aligned} \sigma &= \left(1 - \frac{\alpha+2}{4\alpha}z + \frac{\alpha+1}{8\alpha^2}z^2 - \frac{3\alpha+2}{96\alpha^3}z^3 + O(z^4)\right)(\varepsilon - \varepsilon_m) + \sigma_m \\ \sigma' &= 1 - \frac{(\alpha+1)(\alpha+2)}{4\alpha}z + \frac{(\alpha+1)(2\alpha+1)}{8\alpha^2}z^2 - \\ &- \frac{(3\alpha+1)(3\alpha+2)}{96\alpha^3}z^3 + O(z^4), \quad \alpha \neq 0 \end{aligned} \tag{1.9}$$

## 2. THE TRANSVERSE OSCILLATIONS OF A ROD

We will now consider the forced, plane, transverse oscillations of a thin, uniform, rectilinear rod, which is supported in hinges at its ends. In order to take account of the scattering of energy, we will use the model described above with a power dependence of the absorption coefficient on the amplitude. We will introduce dimensionless parameters by dividing all the linear dimensions and the flexure of the rod by the quantity $\sqrt{F}$, where $F$ is the cross-section area. We will divide the time by $\sqrt{F}/c$, the stresses, as earlier, by the modulus of elasticity $E$ and the per unit length by $E\sqrt{F}$. Here, $c^2 = E/\rho$ is the speed of sound in the rod and $\rho$ is the density. We shall restrict the investigation to weak damping and only take account of the first two terms of expansion (1.9)

$$\sigma = \left(1 - \frac{\alpha+2}{4\alpha}A|\varepsilon - \varepsilon_m|^\alpha\right)(\varepsilon - \varepsilon_m) + \sigma_m \tag{2.1}$$

To be specific, we shall assume that the rod has a rectangular cross-section $h \times b$. Then, using equality (2.1), we obtain

$$\begin{aligned} M &= \frac{J}{r}\left(1 - b\left|\frac{1}{r_m} - \frac{1}{r}\right|^\alpha\right) - b\frac{J}{r_m}\left(\left|\frac{1}{r_m}\right|^\alpha - \left|\frac{1}{r_m} - \frac{1}{r}\right|^\alpha\right) \\ b &= \frac{3(\alpha+2)}{4\alpha(\alpha+3)}A\left(\frac{h}{2}\right)^\alpha \end{aligned} \tag{2.2}$$

where $M$ is the bending moment, $r$ is the radius of curvature of the neutral layer and $J$ is the moment of inertia of the section.

The equation of motion of the rod under the action of a distributed harmonic load $q$ has the form

$$M_{xx} + w_{tt} = q \sin \nu t \tag{2.3}$$

where $w$ is the bending deflection of the rod and differentiation with respect to the axial coordinate $x$ and the time $t$ is denoted by the subscripts $x$ and $t$.

Since weak damping is being considered, we shall assume that its effect on the form of the oscillations can be neglected and confine ourselves to the single-mode approximation of the oscillations of the rod

$$
\begin{aligned}
w(x,t) &= W(t)\sin kx, \quad \frac{1}{r} = -k^2 W\sin kx, \quad q(x) = Q\sin kx \\
M(x,t) &= M_0(t)\sin kx, \quad k = \frac{\pi}{l}
\end{aligned}
\tag{2.4}
$$

where $l$ is the length of the rod.

Substituting relations (2.4) into equalities (2.2) and (2.3) we then multiply them by $\sin kx$ and integrate with respect to $x$ from 0 to $l$. After eliminating the bending moment, the equation of motion can be written in the following form

$$
\begin{aligned}
&W_{tt} + k^4 JW = \xi f(W) + Q\sin\nu t \\
&f(W) = W|W_m - W|^{\alpha} + W_m(|W_m|^{\alpha} - |W_m - W|^{\alpha}) \\
&\xi = k_\alpha AJ\left(\frac{h}{2}\right)^{\alpha}\left(\frac{\pi}{l}\right)^{4+2\alpha}, \quad k_\alpha = \frac{3}{\sqrt{\pi}\alpha(\alpha+3)}\frac{\Gamma(\alpha/2+3/2)}{\Gamma(\alpha/2+1)}
\end{aligned}
\tag{2.5}
$$

The quantity $\xi$ can be regarded as a small parameter. For the chosen method of introducing the dimensionless parameters $(\pi/l)^{4+2\alpha} \ll 1$, and the factor $A$, which is proportional to the absorption coefficient, is small quantity. A graph of $k_\alpha$ against $\alpha$ is shown in Fig. 1. The remaining factors in the expression for $\xi$ have a magnitude of the order of $h^n$, where $n > 1$.

We will consider the oscillations of a rod under the action of a harmonic load of small amplitude, when it can be assumed that $Q = \xi R$. We shall solve Eq. (2.5) by the method of asymptotic averaging [6]. We change to the osculating variables, the amplitude $a$ and the phase angles $\vartheta$

$$
W = a\cos(\nu t + \vartheta)
\tag{2.6}
$$

where $a$ and $\vartheta$ must satisfy the equalities

$$
\begin{aligned}
&a_t = -\delta_e a - \frac{\xi R}{\omega+\nu}\cos\vartheta, \quad \vartheta_t = \omega_e - \nu + \frac{\xi R}{a(\omega+\nu)}\sin\vartheta; \quad \omega^2 = Jk^4 \\
&\delta_e = \frac{\xi}{2\pi a\omega}\int_0^{2\pi} f(a\cos\varphi)\sin\varphi d\varphi = \frac{\xi}{2\pi\omega}a^{\alpha}I_1, \quad I_1 = \frac{2\alpha}{\alpha+2} \\
&\omega_e^2 = k_e = \omega^2 - \frac{\xi}{\pi a}\int_0^{2\pi} f(a\cos\varphi)\cos\varphi d\varphi = \omega^2 - \frac{2\xi}{\pi}a^{\alpha}I_2 \\
&I_2 = 1 - \int_0^{\pi/2}[(1-\cos\varphi)^{\alpha+1}\cos\varphi - \sin^{\alpha+2}\varphi]d\varphi
\end{aligned}
\tag{2.7}
$$

Here $\delta_e$, $k_e$, $\omega_e$ are the decrement, the stiffness and the frequency of natural oscillations of the equivalent linearized system. The frequency of the external load $\nu$ is close to $\omega_e$.

The integral $I_2$ can only be successfully expressed in quadratures for discrete values of $\alpha$; for example, it is equal to $\pi/2$, $\pi/2$, $9\pi/8-2$, $55\pi/16-28/3$ when $\alpha = 0, 1, 2, 4$ respectively. The graph of $I_2(\alpha)$, obtained by numerical integration, is shown in Fig. 1. When $0 < \alpha < 2$, we can assume $I_2 = 1.55$ with an accuracy up to 1.3%.

In the case of steady oscillations, it is necessary to put $a_t = 0$ and $\vartheta_t = 0$. Then, on eliminating $\vartheta$ from the first two equations of (2.7), we obtain

$$
a^2[(\omega_e-\nu)^2(\omega+\nu)^2 + \delta_e^2(\omega+\nu)^2]a^2 = \xi^2 R^2
\tag{2.8}
$$

Apart from quantities of the order of $\xi^2$, relation (2.8) is transformed to the form

$$
a^2[(\omega_e^2-\nu^2)^2 + 4\omega^2\delta_e^2] = Q^2
\tag{2.9}
$$

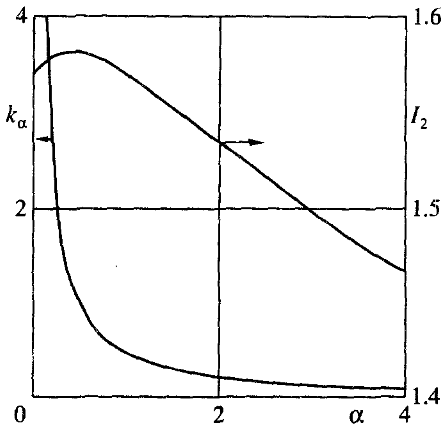


Fig. 1

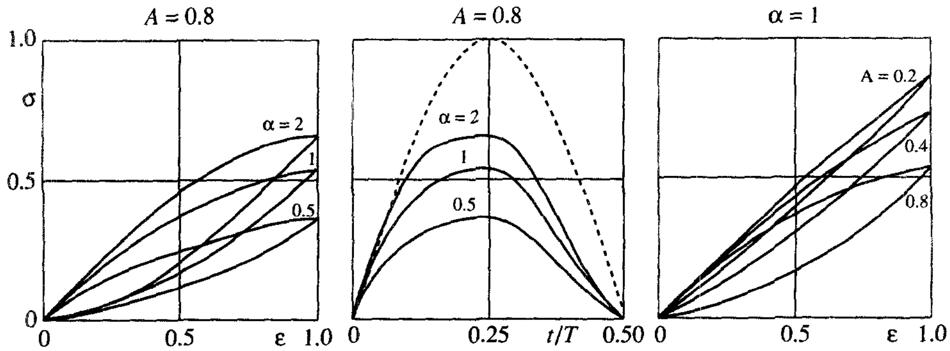


Fig. 2

In order that Eq. (2.9) should simulate the scattering of energy which independent of the excitation frequency ν, instead of the substitution which is usually used, the substitution $(\omega + \nu)2 \approx 4\omega^2$, with the same order of accuracy, is made when approximating the term containing $\delta_e^2$. Using expressions (2.7) for $\delta_e^2$ and $\omega_e^2$, after reduction, we have

$$u^2 = [(1 - I_2 s u^\alpha - p^2)^2 + I_1^2 s^2 u^{2\alpha}]^{-1}$$
$$u = \frac{\omega^2}{Q}a, \quad s = \frac{2}{\pi}Q^\alpha\left(\frac{\xi}{\omega^2}\right)^{\alpha+1}, \quad p = \frac{\nu}{\omega} \tag{2.10}$$

On the basis of equalities (2.10), the expressions for the resonance curve, the skeletal curve and the absorption coefficient will be

$$p = (1 - I_2 s u^\alpha \pm (1/u^2 - I_1^2 s^2 u^{2\alpha})^{1/2})^{1/2}$$
$$p = (1 - I_2 s u^\alpha)^{1/2}, \quad \psi = 2\pi I_1 s u^\alpha \tag{2.11}$$

respectively.

Apart from the asymptotic solution for Eq. (2.5), a numerical solution was also obtained. After the introduction of parameters, which are normalized in a similar way to the parameters in expression (2.10), Eq. (2.5) can be written in the following form

$$\frac{d^2u^*}{d\tau^2} + u^* = \frac{\pi}{2}sf(u^*) + \sin p\tau$$
$$f(u^*) = u^*|u_m^* - u^*|^\alpha + u_m^*(|u_m^*|^\alpha - |u_m^* - u^*|^\alpha), \quad u^* = \frac{\omega^2}{Q}W, \quad \tau = t\omega \tag{2.12}$$

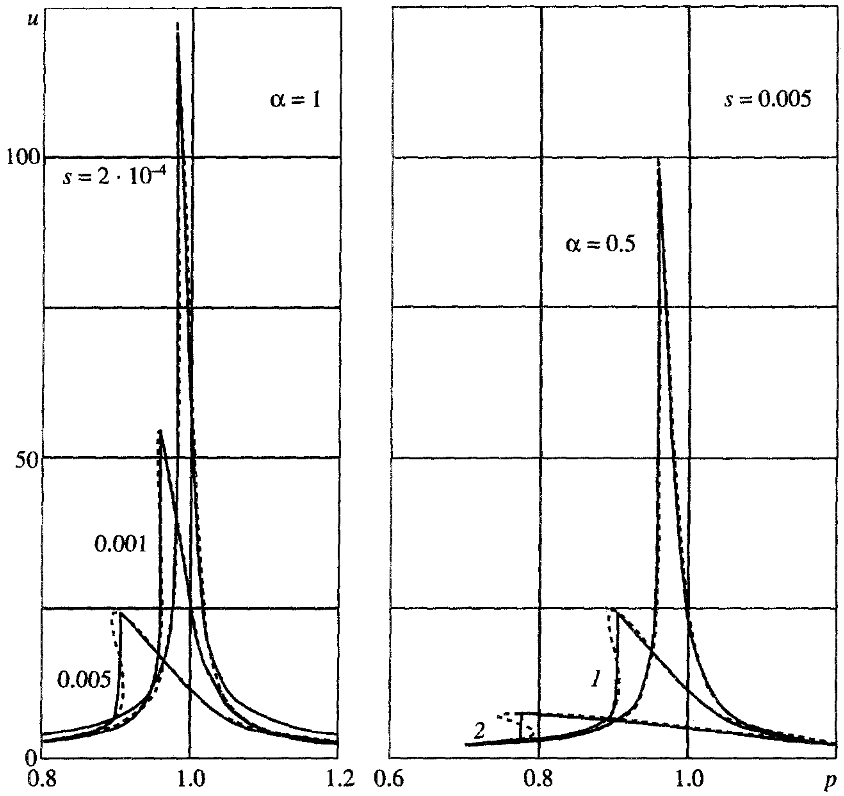


Fig. 3

A load corresponds to $du^* | / d\tau \geq 0$. As in the asymptotic solution, $u$ is the amplitude value of the parameter $u^*$.

Equation (2.12) was reduced to a system of two first-order equations and integrated with zero initial conditions using the fourth-order Runge–Kutta method [7].

The integration step was taken to be $10^{-3}$. In all cases, control calculations with a step size of $10^{-4}$ showed that the results were identical to no less than four significant figures. Steady-state conditions were attained due to the fairly long observation time during which transients could decay.

## 3. RESULTS OF NUMERICAL CALCULATIONS

The hysteresis loops corresponding to a power dependence of the absorption coefficient on the amplitude change not only their shape when the parameters of the model are varied but also their orientation in the $\sigma - \varepsilon$ plane. Loops for different values of $\alpha$ and a fixed parameter $A = 0.8$ are shown on the left in Fig. 2. In this case, the strain varied harmonically with unit amplitude. It is clear that, as $\alpha$ increases, not only does the area of the loop become larger but its angle of inclination to the abscissa axis also increases. The shape of the loop possesses central symmetry with respect to the origin of coordinates and each of its lobes is symmetrical about the middle of the section joining the origin of coordinates to the maximum stress. This shape of loop is fully in accord with relation (1.1). Similar hysteresis loops are also characteristic for a model [4] which is obtained by a completely different route but which can also satisfactorily approximate the power amplitude characteristics of the absorption coefficient. It was found to be close in its structure to the model being considered in the case of weak damping.

The change in the stresses with time is shown in the middle part of Fig. 2 for the same values of the parameters. The time is normalized to the period of the oscillations $T$. The variation in the strains is shown by the dashed curve for comparison. The shape of the curves possesses central symmetry about the point (0.5, 0). At the end of the loading and unloading stages, the stresses change much more slowly that at the start, and almost plane parts with a small curvature are formed on the half waves. The growth in their width as the parameter $\alpha$ is reduced is accompanied by a fall in the amplitude of the stresses. When there is a change of sign of the stresses, a kink is clearly visible an the $\sigma(t)$ curves. The transition from loading to unloading after reaching the maximum stresses occurs more smoothly.

The change in the hysteresis loop when the parameter $A$ is varied while retaining the linear relation between the absorption coefficient and the amplitude, that is, when $\alpha = 1$, is shown on the right-hand side of Fig. 2. Here, the area of the loop becomes larger as $A$ increases but, in this case, its inclination to the horizontal axis does not increase, as in the preceding case, but decreases.

The resonance curves for the steady oscillations of a rod with $\alpha = 1$ and different values of the parameter $s$ are shown to the left in Fig. 3, and the same curves in the case of constant $s$ and different $\alpha$ are shown on the right-hand side of this figure. The solid curves correspond to the numerical solution and dashed curves to the asymptotic solution. Naturally, the equation of motion obtained by numerical integration does not describe the unstable parts of the resonance curves which have been constructed from the asymptotic solution. During the course of the integration, a discontinuous change in the amplitude to the left of the resonance peak is clearly observed. Over the range of variation of the parameters being considered, the asymptotic solution gives sufficiently high accuracy. Its good agreement with the numerical solution is indicative of this.

The shape obtained for the resonance curves is due to the non-linearity of the dynamical system and corresponds to the "soft" characteristic of the restoring force when the rate of increase in the force decreases as the displacement become larger. An increase in the parameters $\alpha$ and $s$ and the intensification of the damping associated with it brings about a reduction in the frequency at which a jump in the magnitude of the amplitude is observed and, also, in the frequency where the peak of the resonance curve is observed. In the case of weak damping, the shape of the resonance peak becomes close to symmetrical.

The proposed procedure enables one to construct models of frequency-independent energy scattering, the use of which does not lead to the accumulation of residual strains. Moreover, the curves of the stresses against the strains corresponding to them do not contain discontinuities in the stresses. It is therefore convenient to use them to describe not only periodic processes but, also, transient wave processes [4].

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